Factorization of a class of Toeplitz + Hankel operators and the A_p-condition

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Abstract

Let $M(\phi) = T(\phi) + H(\phi)$ be the Toeplitz plus Hankel operator acting on $H^p(\mathbb{T})$ with generating function $\phi \in L^\infty(\mathbb{T})$. In a previous paper we proved that $M(\phi)$ is invertible if and only if ϕ admits a factorization $\phi(t) = \phi_-(t)\phi_0(t)$ such that ϕ_- and ϕ_0 and their inverses belong to certain function spaces and such that a further condition formulated in terms of ϕ_- and ϕ_0 is satisfied. In this paper we prove that this additional condition is equivalent to the Hunt-Muckenhoupt-Wheeden condition (or, A_p-condition) for a certain function σ defined on $[-1, 1]$, which is given in terms of ϕ_0. As an application, a necessary and sufficient criteria for the invertibility of $M(\phi)$ with piecewise continuous functions ϕ is proved directly. Fredholm criteria are obtained as well.

1   Introduction

This paper is devoted to continuing the study (started in [2]) of operators of the form

$$M(\phi) \;=\; T(\phi) + H(\phi) \tag{1}$$

acting on the Hardy space $H^p(\mathbb{T})$ where $1 < p < \infty$. Here $\phi \in L^\infty(\mathbb{T})$ is a Lebesgue measurable and essentially bounded function on the unit circle $\mathbb{T}$. The Toeplitz and Hankel operators are defined by

$$T(\phi) : f \mapsto P(\phi f), \qquad H(\phi) : f \mapsto P(\phi(Jf)), \qquad f \in H^p(\mathbb{T}), \qquad (2)$$

where is J the following flip operator,

$$J : f(t) \mapsto t^{-1} f(t^{-1}), \quad t \in \mathbb{T}, \qquad (3)$$

acting on the Lebesgue space $L^p(\mathbb{T})$. The operator P stands for the Riesz projection,

$$P : \sum_{n=-\infty}^{\infty} f_n t^n \mapsto \sum_{n=0}^{\infty} f_n t^n, \quad t \in \mathbb{T}, \qquad (4)$$

which is bounded on $L^p(\mathbb{T})$, $1 < p < \infty$, and whose image is $H^p(\mathbb{T})$. The complex conjugate Hardy space $\overline{H^p(\mathbb{T})}$ is the set of all functions f whose complex conjugate belongs to $H^p(\mathbb{T})$. Moreover, we denote by $L^p_{\text{even}}(\mathbb{T})$ the subspace of $L^p(\mathbb{T})$ consisting of all even functions, i.e., functions f for which $f(t) = f(t^{-1})$.

For $\phi \in L^\infty(\mathbb{T})$ we denote by $L(\phi)$ the multiplication operator acting on $L^p(\mathbb{T})$,

$$L(\phi) \quad : \quad f(t) \mapsto \phi(t) f(t). \qquad (5)$$

Obviously, $T(\phi)$ and $H(\phi)$ can be written as

$$T(\phi) = PL(\phi)P|_{H^p(\mathbb{T})}, \qquad H(\phi) = PL(\phi)JP|_{H^p(\mathbb{T})}.$$

In a previous paper [2] we proved that for $\phi \in L^\infty(\mathbb{T})$ the operator $M(\phi)$ is a Fredholm operator on the space $H^p(\mathbb{T})$ if and only if the functions ϕ admits a certain kind of generalized factorization. Before recalling the underlying definitions, let us state the following simple necessary condition for the Fredholmness of $M(\phi)$ which was also established in [2, Prop. 2.2]. Therein $GL^\infty(\mathbb{T})$ stands for the group of all invertible elements in the Banach algebra $L^\infty(\mathbb{T})$.

Proposition 1.1 *Let $1 < p < \infty$ and $\phi \in L^\infty(\mathbb{T})$. If $M(\phi)$ is Fredholm on $H^p(\mathbb{T})$, then $\phi \in GL^\infty(\mathbb{T})$.*

A function $\phi \in L^\infty(\mathbb{T})$ is said to admit a *weak asymmetric factorization in $L^p(\mathbb{T})$* if it can be written in the form

$$\phi(t) = \phi_-(t) t^\varkappa \phi_0(t), \qquad t \in \mathbb{T}, \qquad (6)$$

such that $\varkappa \in \mathbb{Z}$ and

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(i) $(1 + t^{-1})\phi_- \in \overline{H^p(\mathbb{T})}$, $(1 - t^{-1})\phi_-^{-1} \in \overline{H^q(\mathbb{T})}$,

(ii) $|1 - t|\phi_0 \in L_{\text{even}}^q(\mathbb{T})$, $|1 + t|\phi_0^{-1} \in L_{\text{even}}^p(\mathbb{T})$.

Here $1/p + 1/q = 1$. It was proved in [2, Prop. 3.1] that if a factorization exists, then the *index* $\varkappa$ of the weak asymmetric factorization is uniquely determined and the factors ϕ_- and ϕ_0 are uniquely determined up to a multiplicative constant. (In [2] also the notion of a *weak antisymmetric factorization in* $L^p(\mathbb{T})$ has been introduced. This notion will play no role in the present paper.)

In order to introduce yet another notion, let $\mathcal{R}$ stand for the set of all trigonometric polynomials. Under the assumption that ϕ admits a weak asymmetric factorization in $L^p(\mathbb{T})$ introduce the linear spaces

$$X_1 \;=\; \Big\{\, (1 - t^{-1})f(t) \;:\; f \in \mathcal{R} \,\Big\}, \tag{7}$$

$$X_2 \;=\; \Big\{\, (1 + t^{-1})\phi_0^{-1}(t)f(t) \;:\; f \in \mathcal{R}, \; f(t) = f(t^{-1}) \,\Big\}. \tag{8}$$

It is easy to see that X_1 and X_2 are linear subspaces of $L^p(\mathbb{T})$ and that the space X_1 is dense in $L^p(\mathbb{T})$. Moreover, it has been proved [2, Lemma 4.1(a)] that

$$B \;:=\; L(\phi_0^{-1})(I + J)PL(\phi_-^{-1}) \tag{9}$$

is a well-defined linear (not necessarily bounded) operator acting from X_1 into X_2.

We will call the above factorization (6) of ϕ an *asymmetric factorization in* $L^p(\mathbb{T})$ if in addition to (i) and (ii) the following condition is satisfied:

(iii) The operator $B = L(\phi_0^{-1})(I + J)PL(\phi_-^{-1})$ acting from X_1 into X_2 can be extended by continuity to a linear bounded operator acting on $L^p(\mathbb{T})$.

Clearly, due to the density of X_1 in $L^p(\mathbb{T})$ an equivalent formulation for condition (iii) is the following statement:

(iii*) There exists a constant M such that $\|Bf\|_{L^p(\mathbb{T})} \le M\|f\|_{L^p(\mathbb{T})}$ for all $f \in X_1$.

The main result proved in [1, Thm. 6.4] is the following:

Theorem 1.2 *Let* $1 < p < \infty$ *and* $\phi \in GL^\infty(\mathbb{T})$. *The operator* $M(\phi)$ *is a Fredholm operator on* $H^p(\mathbb{T})$ *if and only if the function* ϕ *admits an asymmetric factorization in* $L^p(\mathbb{T})$. *In this case, the defect numbers are given by*

$$\dim \ker M(\phi) = \max\{0, -\varkappa\}, \qquad \dim \ker M^*(\phi) = \max\{0, \varkappa\}, \tag{10}$$

where $\varkappa$ *is the index of the factorization of* ϕ.

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To formulate the main result of this paper we need the notion of the Hunt-Muckenhoupt-Wheeden condition (or, A_p-condition) with respect to the interval $[-1, 1]$.

Let $1 < p < \infty$, $1/p + 1/q = 1$, and let $\sigma : [-1, 1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. Assume in addition that $\sigma \in L^p[-1, 1]$ and $\sigma^{-1} \in L^q[-1, 1]$. We say that σ satisfies the A_p-condition on $[-1, 1]$ if

$$\sup_I \frac{1}{|I|} \left(\int_I \sigma^p(x)\, dx \right)^{1/p} \left(\int_I \sigma^{-q}(x)\, dx \right)^{1/q} \quad < \quad \infty, \tag{11}$$

where the supremum is taken over all subintervals I of $[-1, 1]$. The length of the interval I is denoted by $|I|$. There is an intimate connection between the A_p-condition and the boundedness of the singular integral operator, which will be stated later on.

The main result of this paper is the following:

Theorem 1.3 *Let* $1 < p < \infty$, $1/p + 1/q = 1$, *and* $\phi \in L^\infty(\mathbb{T})$. *The operator* $M(\phi)$ *is a Fredholm operator on* $H^p(\mathbb{T})$ *if and only if the following conditions are satisfied:*

(a) $\phi \in GL^\infty(\mathbb{T})$.

(b) The function ϕ *admits a weak asymmetric factorization in* $L^p(\mathbb{T})$,

$$\phi(t) \quad = \quad \phi_-(t) t^\varkappa \phi_0(t), \quad t \in \mathbb{T}. \tag{12}$$

(c) The function

$$\sigma(\cos\theta) \quad := \quad |\phi_0^{-1}(e^{i\theta})| \frac{|1 + \cos\theta|^{1/2q}}{|1 - \cos\theta|^{1/2p}} \tag{13}$$

   satisfies the A_p-*condition.*

Moreover, formulas (10) hold in this case.

We note that it is straightforward to prove that condition (b) of the previous theorem implies $\sigma \in L^p[-1, 1]$ and $\sigma^{-1} \in L^q[-1, 1]$.

2 Proof of Theorem 1.3

Before we are able to give the proof of Theorem 1.3 we establish some definitions and auxiliary results.

Let $C^\infty[-1, 1]$ stand for the set of all infinitely differentiable functions $f : [-1, 1] \to \mathbb{C}$, and denote by $C_0^\infty[-1, 1]$ the subspace of all functions $f \in C^\infty[-1, 1]$ such that $f(x)$ and all of its derivatives vanish at the endpoints $x = -1$ and $x = 1$,

$$C_0^\infty[-1, 1] \quad = \quad \left\{ f \in C^\infty[-1, 1] \,:\, f^{(n)}(-1) = f^{(n)}(1) = 0 \text{ for all } n \geq 0 \right\}. \tag{14}$$

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The singular integral operator $S_{[-1,1]}$ is defined by the rule

$$(S_{[-1,1]}f)(x) \;\; = \;\; \frac{1}{\pi i} \int_{-1}^{1} \frac{f(y)}{y-x}\, dy, \qquad x \in [-1,1], \tag{15}$$

where the integral has to be inderstood as the Cauchy principle value. For $f \in C_0^\infty[-1,1]$ the integral exists for each $x \in [-1,1]$. In fact,

$$(S_{[-1,1]}f)(x) \;\; = \;\; \frac{1}{\pi i} \int_{-1}^{1} \frac{f(y)-f(x)}{y-x}\, dy + \frac{f(x)}{\pi i} \ln\left(\frac{1-x}{1+x}\right), \qquad x \in [-1,1]. \tag{16}$$

In particular, $S_{[-1,1]}$ is a well defined linear mapping acting from $C_0^\infty[-1,1]$ into $C^\infty[-1,1]$.

Let $\sigma : [-1,1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. We denote by $L_\sigma^p[-1,1]$ the space consisting of all Lebesgue mesurable functions $f : [-1,1] \to \mathbb{C}$ for which

$$\|f\|_{L_\sigma^p[-1,1]} \;\; := \;\; \left(\int_{-1}^{1} \sigma^p(x)|f(x)|^p\, dx \right)^{1/p} \;\; < \;\; \infty. \tag{17}$$

For certain functions σ it is possible to extend the singular integral operator $S_{[-1,1]}$ as defined above on $C_0^\infty[-1,1]$ by continuity to a linear bounded operator acting on the Banach space $L_\sigma^p[-1,1]$. The criteria is related to the A_p-condition on $[-1,1]$.

The following theorem was established in the case of the real line $\mathbb{R}$ rather than the interval $[-1,1]$ first by Hunt, Muckenhoupt and Wheeden [6]. The theorem itself follows from the results of Coifman and Fefferman [5]. For more information about the A_p condition and the boundedness of the singular integral operators on more general curves we refer [3].

Theorem 2.1 *Let $\sigma : [-1,1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. Assume that $\sigma \in L^p[-1,1]$ and $\sigma^{-1} \in L^q[-1,1]$, where $1 < p < \infty$, $1/p + 1/q = 1$. Then $S_{[-1,1]} : C_0^\infty[-1,1] \to C^\infty[-1,1]$ can be continued by continuity to a linear bounded operator acting on $L_\sigma^p[-1,1]$ if and only if σ satisfies the A_p-condition on $[-1,1]$.*

We remark in connection with the previous theorem that the assumptions that σ is nonzero almost everywhere and that $\sigma \in L^p[-1,1]$ imply that $C_0^\infty[-1,1]$ is a dense linear subspace of $L_\sigma^p[-1,1]$.

The proof of this statement is similar to the proof of Lemma 2.2 below. Let $C^\infty(\mathbb{T})$ stand for the set of all infinitely differentiable functions on $\mathbb{T}$, and let $C_0^\infty(\mathbb{T})$ stand for the space of all $f \in C^\infty(\mathbb{T})$ such that $f(t)$ and all of its derivatives vanish at $t = 1$ and $t = -1$:

$$C_0^\infty(\mathbb{T}) \;\; = \;\; \left\{ f \in C^\infty(\mathbb{T}) \; : \; f^{(n)}(1) = f^{(n)}(-1) = 0 \text{ for all } n \geq 0 \right\}. \tag{18}$$

Let $\varrho : \mathbb{T} \to \mathbb{R}_+$ be a Lebesgue measurable and almost everywhere nonzero function. We denote by $L_\varrho^p(\mathbb{T})$ the space of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\|f\|_{L_\varrho^p(\mathbb{T})} \ := \ \left(\frac{1}{2\pi} \int_0^{2\pi} \varrho^p(e^{i\theta}) |f(e^{i\theta})|^p \, d\theta \right)^{1/p} \ < \ \infty. \tag{19}$$

Let us remark that the dual space to $L_\varrho^p(\mathbb{T})$ can be identified with $L_{\varrho^{-1}}^q(\mathbb{T})$ by means of the bilinear functional

$$\langle g, f \rangle \ := \ \frac{1}{2\pi} \int_0^{2\pi} \overline{g(e^{i\theta})} f(e^{i\theta}) \, d\theta \tag{20}$$

with $f \in L_\varrho^p(\mathbb{T})$, $g \in L_{\varrho^{-1}}^q(\mathbb{T})$, $1 < p < \infty$, $1/p + 1/q = 1$.

Lemma 2.2 *Let $\varrho \in L^p(\mathbb{T})$, $1 < p < \infty$, and assume that ϱ is nonzero almost everywhere. Then $C_0^\infty(\mathbb{T})$ is a dense subspace of $L_\varrho^p(\mathbb{T})$.*

Proof. We introduce the set

$$X \ = \ \left\{ \, \varrho f \ : \ f \in C_0^\infty(\mathbb{T}) \, \right\}. \tag{21}$$

Obviously, $X \subseteq L^p(\mathbb{T})$, which implies that $C_0^\infty(\mathbb{T})$ is a subset of $L_\varrho^p(\mathbb{T})$. The assertion that $C_0^\infty(\mathbb{T})$ is a dense subspace in $L_\varrho^p(\mathbb{T})$ is equivalent to the statement that X is a dense subspace of $L^p(\mathbb{T})$.

We carry out the proof of this statement in several steps. First we prove that the closure of X contains all functions of the form $f = \varrho g$ with $g \in L^\infty(\mathbb{T})$. Indeed, given $g \in L^\infty(\mathbb{T})$ and $\varepsilon > 0$, there are a subset $M \subset \mathbb{T}$ of Lebesgue measure less than ε and a sequence $g_n \in C_0^\infty(\mathbb{T})$ such that $g_n \to g$ uniformly on $\mathbb{T} \setminus M$ and $\|g_n\|_{L^\infty(\mathbb{T})} \leq \|g\|_{L^\infty(\mathbb{T})}$. Now we can estimate

$$\|\varrho g - \varrho g_n\|_{L^p(\mathbb{T})} \ \leq \ 2 \, \|\varrho\|_{L^p(M)} \|g\|_{L^\infty(\mathbb{T})} + \|\varrho\|_{L^p(\mathbb{T})} \|g_n - g\|_{L^\infty(\mathbb{T} \setminus M)}.$$

Since $\|\varrho\|_{L^p(M)} \to 0$ as $\varepsilon \to 0$ the assertion follows easily.

Next we prove that $L^\infty(\mathbb{T})$ is contained in the closure of X. Indeed, given $f \in L^\infty(\mathbb{T})$ we introduce the elements $g_\varepsilon = \varrho_\varepsilon f$ where

$$\varrho_\varepsilon(t) \ = \ \begin{cases} \varrho^{-1}(t) & \text{if } 0 < \varrho^{-1}(t) \leq \varepsilon^{-1}, \\ 0 & \text{if } \varrho^{-1}(t) > \varepsilon^{-1}. \end{cases}$$

Obviously, $\varrho_\varepsilon \in L^\infty(\mathbb{T})$ and hence $g_\varepsilon \in L^\infty(\mathbb{T})$. Now we estimate

$$\|\varrho g_\varepsilon - f\|_{L^p(\mathbb{T})} \ = \ \|(\varrho \varrho_\varepsilon - 1) f\|_{L^p(\mathbb{T})} \ \leq \ \|\varrho \varrho_\varepsilon - 1\|_{L^p(\mathbb{T})} \|f\|_{L^\infty(\mathbb{T})}.$$

The function $1 - \varrho\varrho_\varepsilon$ is equal to the characteristic function of

$$K_\varepsilon = \left\{ t \in \mathbb{T} : \varrho^{-1}(t) > \varepsilon^{-1} \right\}.$$

Since ϱ is nonzero almost everywhere, the Lebesgue measure of K_ε tends to zero as $\varepsilon \to 0$. Hence $\|\varrho\varrho_\varepsilon - 1\|_{L^p(\mathbb{T})} \to 0$ as $\varepsilon \to 0$. It follows that ϱg_ε approximates f.

Finally, it we note that $L^\infty(\mathbb{T})$ is dense in $L^p(\mathbb{T})$. $\qquad\square$

Let $Q = I - P$ stand for the complementary projection to the Riesz projection P. Moreover, define the operators

$$G = \frac{1}{2}(I + J)(P - Q)(I - J), \tag{22}$$

$$G^* = \frac{1}{2}(I - J)(P - Q)(I + J). \tag{23}$$

We think of G and G^* as linear mappings acting from $C_0^\infty(\mathbb{T})$ into $C^\infty(\mathbb{T})$. Notice in this connection that P and Q map $C_0^\infty(\mathbb{T})$ into $C^\infty(\mathbb{T})$.

Proposition 2.3 *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Assume that $\phi \in GL^\infty(\mathbb{T})$ admits a weak asymmetric factorization $\phi(t) = \phi_-(t)t^\varkappa\phi_0(t)$ in $L^p(\mathbb{T})$. Then the following is equivalent:*

(i) *The operator $B := L(\phi_0^{-1})(I + J)PL(\phi_-^{-1}) : X_1 \to X_2$ can be continued by continuity to a linear bounded operator acting on $L^p(\mathbb{T})$.*

(iii) *The operator $G^* : C_0^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ can be continued by continuity to a linear bounded operator acting on $L_{\omega^{-1}}^q(\mathbb{T})$.*

(iii) *The operator $G : C_0^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ can be continued by continuity to a linear bounded operator acting on $L_\omega^p(\mathbb{T})$.*

Therein $\omega(t) = |\phi_0^{-1}(t)|$, and X_1 and X_2 are defined by (7) and (8), respectively.

Proof. (i)$\Leftrightarrow$(ii): Assertion (i) is equivalent to the fact that there exists a constant $M > 0$ such that

$$\|L(\phi_0^{-1})(I + J)PL(\phi_-^{-1})f\|_{L^p(\mathbb{T})} \leq M\|f\|_{L^p(\mathbb{T})}$$

for all $f \in X_1$. From the definition of $L_\omega^p(\mathbb{T})$ it follows that the last inequality can be rewritten as

$$\|(I + J)PL(\phi_-^{-1})f\|_{L_\omega^p(\mathbb{T})} \leq M\|f\|_{L^p(\mathbb{T})}.$$

Since $C_0^\infty(\mathbb{T})$ is dense in $L_{\omega^{-1}}^q(\mathbb{T})$ by Lemma 2.2, we obtain that this is in turn equivalent to the statement that

$$|\langle g, (I+J)PL(\phi_-^{-1})f\rangle| \leq M\|g\|_{L_{\omega^{-1}}^q(\mathbb{T})}\|f\|_{L^p(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$ and all $f \in X_1$. Next notice that

$$\langle g, (I+J)PL(\phi_-^{-1})f\rangle = \langle P(I+J)g, L(\phi_-^{-1})f\rangle = \langle L((\phi_-^{-1})^*)P(I+J)g, f\rangle$$

by noting that $L(\phi_-^{-1})f \in L^q(\mathbb{T})$. Hence the above is equivalent to the statement that

$$|\langle L((\phi_-^{-1})^*)P(I+J)g, f\rangle| \leq M\|g\|_{L_{\omega^{-1}}^q(\mathbb{T})}\|f\|_{L^p(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$ and all $f \in X_1$. Since X_1 is dense in $L^p(\mathbb{T})$ we can reformulate this by saying that

$$\|L((\phi_-^{-1})^*)P(I+J)g\|_{L^q(\mathbb{T})} \leq M\|g\|_{L_{\omega^{-1}}^q(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$. Because $\phi_-^{-1}(t) = \phi_0(t)\phi^{-1}(t)$ the latter can be rewritten as

$$\|P(I+J)g\|_{L_{\omega^{-1}}^q(\mathbb{T})} \leq M\|g\|_{L_{\omega^{-1}}^q(\mathbb{T})}.$$

Since $\omega(t) = \omega(t^{-1})$ the operator $(I+J)$ is bounded on $L_{\omega^{-1}}^q(\mathbb{T})$. Moreover, since $P - Q = 2P - I$ we can conclude that the latter is equivalent to

$$\|(P-Q)(I+J)g\|_{L_{\omega^{-1}}^q(\mathbb{T})} \leq M\|g\|_{L_{\omega^{-1}}^q(\mathbb{T})} \tag{24}$$

for all $g \in C_0^\infty(\mathbb{T})$. Noting that

$$G^* = \frac{1}{2}(I-J)(P-Q)(I+J) = (P-Q)(I+J)$$

completes the proof of (i)$\Leftrightarrow$(ii).

(ii)$\Leftrightarrow$(iii): Since $C_0^\infty(\mathbb{T})$ is dense in both $L_\omega^p(\mathbb{T})$ and $L_{\omega^{-1}}^q(\mathbb{T})$ by Lemma 2.2, it is easily seen that (ii) is equivalent to the statement that

$$|\langle G^*g, f\rangle| \leq M\|g\|_{L_\omega^p(\mathbb{T})}\|f\|_{L_{\omega^{-1}}^q(\mathbb{T})}$$

for all $f, g \in C_0^\infty(\mathbb{T})$. Moreover, (iii) is equivalent to the statement that

$$|\langle g, Gf\rangle| \leq M\|g\|_{L_\omega^p(\mathbb{T})}\|f\|_{L_{\omega^{-1}}^q(\mathbb{T})}$$

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for all $f, g \in C_0^\infty(\mathbb{T})$. Since $\langle G^* g, f \rangle = \langle g, Gf \rangle$ the result follows. $\qquad\square$

Let $C_{\text{even}}^\infty(\mathbb{T})$ stand for the space of all functions $f \in C^\infty(\mathbb{T})$ which are even, i.e., for which $f(t) = f(t^{-1})$, $t \in \mathbb{T}$. Moreover, introduce the operators

$$U \quad : \quad \hat{f}(x) \in C_0^\infty[-1, 1] \mapsto f(e^{i\theta}) := \hat{f}(\cos\theta) \in C_0^\infty(\mathbb{T}), \tag{25}$$

$$V \quad : \quad f(t) \in C_{\text{even}}^\infty(\mathbb{T}) \mapsto \hat{f}(\cos\theta) := f(e^{i\theta}) \in C^\infty[-1, 1], \tag{26}$$

and let

$$\chi(e^{i\theta}) \quad = \quad \begin{cases} 1 & \text{if } 0 < \theta < \pi, \\ -1 & \text{if } \pi < \theta < 2\pi. \end{cases} \tag{27}$$

Clearly, the image of U is the set of even functions defined on $\mathbb{T}$.

Proposition 2.4 *For $\hat{f} \in C_0^\infty[-1, 1]$ we have*

$$S_{[-1,1]}\hat{f} \quad = \quad \frac{1}{2} V L((1 + t^{-1})^{-1}) G L(\chi(1 + t^{-1})) U \hat{f}. \tag{28}$$

Proof.   Given $\hat{f} \in C_0^\infty[-1, 1]$, we introduce the functions

$$f = L(\chi(1 + t^{-1})) U \hat{f}, \quad g = \frac{1}{2} G f, \quad \hat{g} = V L((1 + t)^{-1}) g.$$

Notice that

$$G = \frac{1}{2}(I + J)(P - Q)(I - J) = (P - Q)(I - J).$$

Hence $g(t) = t^{-1} g(1/t)$ and it follows that $L((1 + t^{-1})^{-1})g$ is an even function. Moreover, it is easily seen that $Jf = -f$ whence it follows that $g = \frac{1}{2} G f = (P - Q)f$. It is well known that the singular integral operator $S = P - Q$ on $\mathbb{T}$ can be written as

$$(Sf)(t) \quad = \quad \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(s) - f(t)}{s - t} \, ds + f(t)$$

for functions $f \in C^\infty(\mathbb{T})$. From this we deduce the relations

$$f(e^{i\theta}) \quad = \quad (1 + e^{-i\theta}) \chi(e^{i\theta}) \hat{f}(\cos\theta),$$

$$g(e^{i\theta}) \quad = \quad \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\varphi}) - f(e^{i\theta})}{1 - e^{i(\theta - \varphi)}} \, d\varphi + f(e^{i\theta}),$$

$$\hat{g}(\cos\theta) \quad = \quad (1 + e^{-i\theta})^{-1} g(e^{i\theta}).$$

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We split the integral appearing in the second equation into two parts integrating on $[0, \pi]$ and $[-\pi, 0]$, respectively, and make a change of variables $\varphi \mapsto -\varphi$ in the second integral. This gives

$$g(e^{i\theta}) = \frac{1}{\pi} \int_0^\pi \left(\frac{f(e^{i\varphi}) - f(e^{i\theta})}{1 - e^{i(\theta-\varphi)}} + \frac{f(e^{-i\varphi}) - f(e^{i\theta})}{1 - e^{i(\theta+\varphi)}} \right) d\varphi + f(e^{i\theta}).$$

Since $f(e^{-i\varphi}) = -e^{i\varphi} f(e^{i\varphi})$ and since

$$\frac{1}{1 - e^{i(\theta-\varphi)}} - \frac{e^{i\varphi}}{1 - e^{i(\theta+\varphi)}} = \frac{(1 + e^{-i\theta})(e^{i\varphi} - 1)}{2(\cos\varphi - \cos\theta)},$$

$$\frac{1}{1 - e^{i(\theta-\varphi)}} + \frac{1}{1 - e^{i(\theta+\varphi)}} = \frac{e^{i\varphi} + e^{-i\varphi} - 2e^{-i\theta}}{2(\cos\varphi - \cos\theta)},$$

it follows that

$$g(e^{i\theta}) = \frac{1}{\pi} \int_0^\pi \left(\frac{(1 + e^{-i\theta})(e^{i\varphi} - 1)f(e^{i\varphi})}{2(\cos\varphi - \cos\theta)} - \frac{(e^{i\varphi} - e^{-i\varphi})f(e^{i\theta})}{2(\cos\varphi - \cos\theta)} \right) d\varphi$$

$$- \frac{f(e^{i\theta})}{\pi} \int_0^\pi \frac{e^{-i\varphi} - e^{-i\theta}}{\cos\varphi - \cos\theta} d\varphi + f(e^{i\theta}).$$

If we assume $0 < \theta < \pi$, we obtain

$$g(e^{i\theta}) = \frac{1}{\pi} \int_0^\pi \frac{(1 + e^{-i\theta})(e^{i\varphi} - e^{-i\varphi})(\hat{f}(\cos\varphi) - \hat{f}(\cos\theta))}{2(\cos\varphi - \cos\theta)} d\varphi$$

$$- \frac{f(e^{i\theta})}{\pi i} \int_0^\pi \frac{\sin\varphi - \sin\theta}{\cos\varphi - \cos\theta} d\varphi.$$

The first integral is equal to $(1 + e^{-i\theta})$ times

$$\frac{i}{\pi} \int_0^\pi \frac{(\hat{f}(\cos\varphi) - \hat{f}(\cos\theta))\sin\varphi}{\cos\varphi - \cos\theta} d\varphi = \frac{1}{\pi i} \int_{-1}^1 \frac{\hat{f}(y) - \hat{f}(\cos\theta)}{y - \cos\theta} dy.$$

The second integral equals $f(e^{i\theta})$ times

$$\frac{1}{\pi i} \int_0^\pi \cot\left(\frac{\varphi + \theta}{2} \right) d\varphi = \left[\frac{2}{\pi i} \ln \sin \left(\frac{\varphi + \theta}{2} \right) \right]_{\varphi=0}^\pi = \frac{1}{\pi i} \ln \left(\frac{1 - \cos\theta}{1 + \cos\theta} \right).$$

Putting the pieces together we obtain

$$g(e^{i\theta}) = \frac{(1 + e^{-i\theta})}{\pi i} \int_{-1}^1 \frac{\hat{f}(y) - \hat{f}(\cos\theta)}{y - \cos\theta} dy + \frac{f(e^{i\theta})}{\pi i} \ln \left(\frac{1 - \cos\theta}{1 + \cos\theta} \right).$$

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Note that the above assumption $0 < \theta < \pi$ is not an essential restriction since $\hat{g}$ is determined by the formula

$$\hat{g}(\cos\theta) = (1 + e^{-i\theta})^{-1} g(e^{i\theta}), \qquad 0 < \theta < \pi.$$

It follows that

$$\hat{g}(x) = \frac{1}{\pi i} \int_{-1}^{1} \frac{\hat{f}(y) - \hat{f}(x)}{y - x}\, dy + \frac{\hat{f}(x)}{\pi i} \ln\left(\frac{1-x}{1+x}\right).$$

This is equal to the singular integral operator $S_{[-1,1]}$ applied to the function $\hat{f}$. Hence $\hat{g} = S_{[-1,1]}\hat{f}$ which is the assertion. $\qquad\square$

Now we are able to present the proof of Theorem 1.3.

Proof of Theorem 1.3.   It is obvious from Proposition 1.1 and Theorem 1.2 that the Fredholmness of $M(\phi)$ implies assertions (a) and (b).

Hence it is sufficient to prove the following. If conditions (a) and (b) are fulfilled, then $M(\phi)$ is Fredholm if and only if σ satisfies the A_p-condition.

Under these assumptions we deduce from Theorem 1.2 that the Fredholmness of $M(\phi)$ on $H^p(\mathbb{T})$ is equivalent to the existence of a bounded continuation of the operator $B = L(\phi_0^{-1})(I + J)PL(\phi_-^{-1}) : X_1 \to X_2$ on $L^p(\mathbb{T})$. We apply Proposition 2.3 and see that this existence is equivalent to the condition that

$$\|Gg\|_{L_\omega^p(\mathbb{T})} \leq M \|g\|_{L_\omega^p(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$ where $\omega(t) := |\phi_0^{-1}(t)|$.

Next we claim that this, in turn, is equivalent to the condition that

$$\|Gg\|_{L_\omega^p(\mathbb{T})} \leq M \|g\|_{L_\omega^p(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$ for which $Jg = -g$. In order to prove the non-trivial part of this equivalence, we decompose an arbitraryly given $g \in C_0^\infty(\mathbb{T})$ into $g = g_1 + g_2$ where $g_1 = \frac{1}{2}(I + J)g$ and $g_2 = \frac{1}{2}(I - J)g$. The function g_1 lies in the kernel of G while $g_2(t) = (g(t) - t^{-1}g(t^{-1}))/2$ belongs to $C_0^\infty(\mathbb{T})$ and satisfies the relation $Jg_2 = -g_2$. Moreover, since ω is an even function, the operator $(I - J)/2$ is bounded on $L_\omega^p(\mathbb{T})$. We obtain the estimate

$$\|Gg\|_{L_\omega^p(\mathbb{T})} = \|Gg_2\|_{L_\omega^p(\mathbb{T})} \leq M \|g_2\|_{L_\omega^p(\mathbb{T})} \leq M \|g\|_{L_\omega^p(\mathbb{T})},$$

proves the claim.

Next we remark that the operator $L(\chi(1 + t^{-1}))U$ maps the space $C_0^\infty[-1,1]$ onto the subspace of functions $g \in C_0^\infty(\mathbb{T})$ satisfying $Jg = -g$. This allows us to make the substitution $g = L(\chi(1 + t^{-1}))Uf$ with $f \in C_0^\infty[-1,1]$. We obtain the equivalent estimate

$$\|GL(\chi(1 + t^{-1}))Uf\|_{L_\omega^p(\mathbb{T})} \leq M \|L(\chi(1 + t^{-1}))Uf\|_{L_\omega^p(\mathbb{T})}$$

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for all $f \in C_0^\infty[-1, 1]$. Clearly, the last estimate can be written in the form

$$\|VL((1+t^{-1})^{-1})GL(\chi(1+t^{-1})Uf\|_{L^p_\sigma[-1,1]} \leq M \|f\|_{L^p_\sigma[-1,1]}$$

for all $f \in C_0^\infty[-1, 1]$, where

$$\sigma(\cos\theta) = \frac{\omega(e^{i\theta})|1 + e^{-i\theta}|}{\sqrt{2}\,|\sin\theta|^{1/p}} = |\phi_0^{-1}(e^{i\theta})|\frac{|1+\cos\theta|^{1/2q}}{|1-\cos\theta|^{1/2p}}.$$

Along with Proposition 2.4 and Theorem 2.1 this completes the proof. □

3 Application to piecewise continuous functions

We now apply the previous results in order to obtain necessary and sufficient conditions for the operator $M(\phi)$ to be invertible or Fredholm on $H^p(\mathbb{T})$ for piecewise continuous functions ϕ with finitely many jumps. These results have already been established in [2] (and in [1] for the case $p = 2$). The proofs given in [1, 2] relies on the results establish in [7] by help of Banach algebra methods. The proof which we will give here is more direct and relies entirely on the factorization methods developed here and in [2] in connection with the A_p-condition.

We restrict to piecewise continuous functions with a *finite* number of discontinuities because these functions can be written in a convenient manner which is useful in many instances. It is well known that any piecewise continuous and nonvanishing function with a finite number of discontinuities at the points $\theta_1, \ldots, \theta_R$ can be written as a product

$$\phi(e^{i\theta}) = b(e^{i\theta})\prod_{r=1}^{R} t_{\beta_r}(e^{i(\theta-\theta_r)}) \tag{29}$$

where b is a nonvanishing continuous function and

$$t_\beta(e^{i\theta}) = \exp(i\beta(\theta - \pi)), \qquad 0 < \theta < 2\pi. \tag{30}$$

Notice that the parameters $\beta_1, \ldots, \beta_R$ in this formula are uniquely determined up to an additive integer. In fact,

$$\frac{\phi(e^{i\theta_r} - 0)}{\phi(e^{i\theta_r} + 0)} = \frac{t_{\beta_r}(1-0)}{t_{\beta_r}(1+0)} = \exp(2\pi i\beta_r). \tag{31}$$

Moreover, the formula

$$t_{\beta+n}(t) = (-t)^n t_\beta(t), \quad t \in \mathbb{T}, \tag{32}$$

holds for $n \in \mathbb{Z}$.

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The parameters in the representation (29) are useful to decide Fredholmness and invertibility. For example, the Toeplitz operators $T(\phi)$ with a piecewise continuous function ϕ with finitely many jump discontinuities is invertible on $H^p(\mathbb{T})$ if and only if the function ϕ can be represented in the form (29) with $-1/q < \operatorname{Re} \beta_r < 1/p$ and the winding number of b equal to zero. If the operator is Fredholm with Fredholm index $\varkappa$, then the zero is replaced by $-\varkappa$. ([4], Chapter 6)

Before stating the analogue of this result for Toeplitz plus Hankel operators $M(\phi)$ we have to establish the following auxiliary result.

Lemma 3.1 *Let* $1 < p < \infty$, $1/p + 1/q = 1$, $-1 = x_0 < x_1 < \ldots < x_R < x_{R+1} = 1$ *and*

$$\sigma(x) = \prod_{r=0}^{R+1} |x - x_r|^{\alpha_r} \tag{33}$$

If $-1/p < \alpha_r < 1/q$ *for each* $0 \leq r \leq R+1$, *then* $\sigma \in L^p[-1,1]$, $\sigma^{-1} \in L^q[-1,1]$ *and* σ *satisfies the* A_p-*condition.*

Proof.   This can be verified straightforwardly.   □

The promised result is the following:

Theorem 3.2 *Let* $1 < p < \infty$, $1/p + 1/q = 1$. *Suppose that* ϕ *has finitely many jump discontinuities. Then* $M(\phi)$ *on is Fredholm on* $H^p(\mathbb{T})$ *if and only if* ϕ *can be written in the form*

$$\phi(e^{i\theta}) = b(e^{i\theta}) t_{\beta^+}(e^{i\theta}) t_{\beta^-}(e^{i(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta_r^+}(e^{i(\theta-\theta_r)}) t_{\beta_r^-}(e^{i(\theta+\theta_r)}) \tag{34}$$

where b *is a continuous nonvanishing function on* $\mathbb{T}$, *the numbers* $\theta_1, \ldots, \theta_R \in (0, \pi)$ *are distinct, and*

(i) $-1/q < \operatorname{Re}(\beta_r^+ + \beta_r^-) < 1/p$ *for each* $1 \leq r \leq R$,

(ii) $-1/2 - 1/2q < \operatorname{Re}\beta^+ < 1/2p$ *and* $-1/2q < \operatorname{Re}\beta^- < 1/2 + 1/2p$.

Moreover, in this case,

$$\dim \ker M(\phi) = \max\{0, -\operatorname{wind}(b)\}, \qquad \dim \ker M^*(\phi) = \max\{0, \operatorname{wind}(b)\}. \tag{35}$$

Proof.   In the first step we prove that $M(\psi)$ is a Fredholm operator on $H^p(\mathbb{T})$ with Fredholm index zero if ψ is of the form

$$\psi(e^{i\theta}) = t_{\beta^+}(e^{i\theta}) t_{\beta^-}(e^{i(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta_r^+}(e^{i(\theta-\theta_r)}) t_{\beta_r^-}(e^{i(\theta+\theta_r)}) \tag{36}$$

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and the parameters satisfy the conditions (i) and (ii). In regard to Theorem 1.3 it suffices to construct a weak asymmetric factorization of ψ and to prove that the corresponding weight σ satisfies the A_p-condition. For this purpose we introduce the functions

$$\eta_\beta(t) = (1-t)^\beta, \qquad \xi_\beta(t) = (1-t^{-1})^\beta.$$

Notice that $t_\beta(t) = \eta_\beta(t)\xi_{-\beta}(t)$. Then we can factor $\psi(t) = \psi_-(t)\psi_0(t)$ with

$$
\begin{aligned}
\psi_-(e^{i\theta}) &= \xi_{-2\beta^+}(e^{i\theta})\xi_{-2\beta^-}(e^{i(\theta-\pi)}) \\
&\quad \times \prod_{r=1}^{R} \xi_{-\beta_r^+-\beta_r^-}(e^{i(\theta-\theta_r)})\xi_{-\beta_r^+-\beta_r^-}(e^{i(\theta+\theta_r)}), \\
\psi_0(e^{i\theta}) &= \eta_{\beta^+}(e^{i\theta})\xi_{\beta^+}(e^{i\theta})\eta_{\beta^-}(e^{i(\theta-\pi)})\xi_{\beta^-}(e^{i(\theta-\pi)}) \\
&\quad \times \prod_{r=1}^{R} \eta_{\beta_r^+}(e^{i(\theta-\theta_r)})\xi_{\beta_r^+}(e^{i(\theta+\theta_r)})\eta_{\beta_r^-}(e^{i(\theta+\theta_r)})\xi_{\beta_r^-}(e^{i(\theta-\theta_r)}).
\end{aligned}
$$

Because of conditions (i) and (ii), it can be checked straightforwardly that the function

$$
\begin{aligned}
(1+e^{-i\theta})\psi_-(e^{i\theta}) &= \xi_{-2\beta^+}(e^{i\theta})\xi_{-2\beta^-+1}(e^{i(\theta-\pi)}) \\
&\quad \times \prod_{r=1}^{R} \xi_{-\beta_r^+-\beta_r^-}(e^{i(\theta-\theta_r)})\xi_{-\beta_r^+-\beta_r^-}(e^{i(\theta+\theta_r)})
\end{aligned}
$$

belongs to $\overline{H^p(\mathbb{T})}$ and the function

$$
\begin{aligned}
(1-e^{-i\theta})\psi_-^{-1}(e^{i\theta}) &= \xi_{2\beta^++1}(e^{i\theta})\xi_{2\beta^-}(e^{i(\theta-\pi)}) \\
&\quad \times \prod_{r=1}^{R} \xi_{\beta_r^++\beta_r^-}(e^{i(\theta-\theta_r)})\xi_{\beta_r^++\beta_r^-}(e^{i(\theta+\theta_r)})
\end{aligned}
$$

belongs to $\overline{H^q(\mathbb{T})}$. From the fact that ψ_0 is even and that $\psi_0(t) = \psi_-(t)^{-1}\psi(t)$, it follows that the function ψ_0 fulfills all the necessary conditions in regard to a weak asymmetric factorization. Hence $\psi(t) = \psi_-(t)\psi_0(t)$ is indeed a weak asymmetric factorization with index zero.

In order to calculate the corresponding weight function (13) consider

$$
\begin{aligned}
\psi_0(e^{i\theta}) &= |1-e^{i\theta}|^{2\beta^+}|1+e^{i\theta}|^{2\beta^-} \\
&\quad \times \prod_{r=1}^{R} |1-e^{i(\theta-\theta_r)}|^{\beta_r^++\beta_r^-}|1-e^{i(\theta+\theta_r)}|^{\beta_r^++\beta_r^-} t_{\frac{\beta_r^+-\beta_r^-}{2}}(e^{i(\theta-\theta_r)})t_{\frac{\beta_r^--\beta_r^+}{2}}(e^{i(\theta+\theta_r)}),
\end{aligned}
$$

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and observe that $|1 - e^{i\theta}| = (2 - 2\cos\theta)^{1/2} = 2|\sin(\frac{\theta}{2})|$ and $2\sin\frac{\theta - \theta_r}{2}\sin\frac{\theta + \theta_r}{2} = \cos\theta_r - \cos\theta$. Hence

$$\psi_0^{-1}(e^{i\theta}) = \sigma_0(\cos\theta)(1 - \cos\theta)^{-\beta^+}(1 + \cos\theta)^{-\beta^-}\prod_{r=1}^{R}|\cos\theta - \cos\theta_r|^{-\beta_r^+ - \beta_r^-},$$

where $\sigma_0(x) \in GL^\infty(\mathbb{T})$ is a functions which comes from collecting the terms $t_{\frac{\beta_r^+ - \beta_r^-}{2}}(e^{i(\theta - \theta_r)})$, $t_{\frac{\beta_r^- - \beta_r^+}{2}}(e^{i(\theta + \theta_r)})$ and certain constants. It follows that σ evaluates to

$$\sigma(x) = |\sigma_0(x)|(1 - x)^{-\operatorname{Re}\beta^+ - 1/2p}(1 + x)^{-\operatorname{Re}\beta^- + 1/2q}\prod_{r=1}^{R}|x - \cos\theta_r|^{-\operatorname{Re}\beta_r^+ - \operatorname{Re}\beta_r^-}. \quad (37)$$

It suffices to apply Lemma 3.1 in order to see that σ satisfies the A_p-condition.

In the second step we prove that $M(\phi)$ is a Fredholm operator if ϕ is given by (34) with conditions (i) and (ii) being fulfilled and if the function b is continuous and nonvanishing. We can write $\phi(t) = b(t)\psi(t)$ where ψ is as above. From well-known identities for Toeplitz and Hankel operators,

$$T(\phi) = T(b)T(\psi) + H(b)H(\widetilde{\psi}),$$

$$H(\phi) = T(b)H(\psi) + H(b)T(\widetilde{\psi}),$$

where $\widetilde{\psi}(t) = \psi(t^{-1})$, it follows that

$$M(\phi) = T(b)M(\psi) + H(b)M(\widetilde{\psi})$$

Under the assumption on b the operator $H(b)$ is compact and the operator $T(b)$ is Fredholm with Fredholm index equal to $-\operatorname{wind}(b)$. Since we have just proved that $M(\psi)$ is Fredholm with Fredholm index zero, it follows that $M(\phi)$ is Fredholm with Fredholm index equal to $-\operatorname{wind}(b)$.

Hence we have proved the "if" part of the theorem and also computed the Fredholm index of $M(\phi)$. Now we apply Theorem 1.2 with formula (10). This formula implies that the Fredholm index is equal to $-\varkappa$, where $\varkappa$ is the index of the asymmetric factorization of ϕ. Hence $\varkappa = \operatorname{wind}(b)$ and formula (35) follows.

In the last step we are going to prove the "only if" part of the theorem. It is settled by a well-known perturbation argument. Suppose that $M(\phi)$ is a Fredholm operator with index $\varkappa$, say. We conclude from Proposition 1.1 that $\phi \in GL^\infty(\mathbb{T})$. Since ϕ has only a finite number of jump discontinuities, this implies that ϕ can be written in the form (29) or in the form (34) if we put some of the β-parameters equal to zero if necessary. Therein the function b is continuous and nonvanishing on $\mathbb{T}$. Moreover, due to formula (31) and (32) we can choose the β-parameters to satisfy the conditions

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(i*) $-1/q < \operatorname{Re}(\beta_r^+ + \beta_r^-) \leq 1/p$ for each $1 \leq r \leq R$,

(ii*) $-1/2 - 1/2q < \operatorname{Re}\beta^+ \leq 1/2p$ and $-1/2q < \operatorname{Re}\beta^- \leq 1/2 + 1/2p$.

Assume contrary to what we want to prove, namely, that in at least one instance we have equality in the above conditions (i*) and (ii*). We are going to perturbate the β-parameters (and thus the function ϕ) in two different ways in order to arrive at a contradiction. Remark that since $M(\phi)$ is assumed to be Fredholm, the Fredholm index is constant with respect to any small perturbation. We first perturbate by replacing all β-parameters by $\beta - \varepsilon$ where $\varepsilon > 0$ is sufficiently small. This turns the conditions (i*) and (ii*) into (i) and (ii). Applying the "if" part of the theorem with the formula for the Fredholm index, it follows that $\varkappa = -\operatorname{wind}(b)$. In the second perturbation we do the same substitution except for one of the instances of equality in (i*) and (ii*) where we replace the corresponding $\beta^{\pm}$ by $\beta^{\pm} - 1 + \varepsilon$, or, the corresponding $\beta_r^+ + \beta_r^-$ by $\beta_r^+ + \beta_r^- - 1 + \varepsilon$, respectively. Moreover, we have to replace $b(t)$ by $t\,b(t)$ times a certain constant due to formula (32). This is again a small perturbation of ϕ, which leaves the Fredholm index unchanged. The corresponding parameters fulfill (i) and (ii), but we obtain $\varkappa = -\operatorname{wind}(t\,b(t)) = -1 - \operatorname{wind}(b)$ contradicting the above formula. This completes the proof of the "only if" part. $\qquad\square$

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